# Why Does a Hula Hoop Stay Up?

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## 1 Introduction

Throughout the civilizations, we have found many creative uses for hoops. Made of reeds, vines, stiff grasses or wood, they have been used in ritual dances, exercises, and as a child's game. There are depictions of children playing with them in ancient Egyptian art, and a Greek krater painted around 500 B.C.E. portrays Ganymede, the most beautiful of mortals, running with a hoop. Hoops were also used in Medieval times. For example, "hooping" became a popular sport in England around the 14th century, until doctors at the time started blaming it for practically everything, from back trouble to heart attacks. In more modern times, hoops made of bamboo were commonly used as exercise equipment in 20th century Australian schools and homes. In 1957, Joan Anderson brought one with her when she immigrated from Australia to California. She recognized its potential, coined the term "hula hoop" for the resemblance between the hip motion needed to keep the hoop up and the Hawaiian hula dance, and started introducing it to her friends. One such "friend", the co-founder of the toy company Wham-O, stole her idea and manufactured the modern plastic hula hoops that became so popular in America.

While many of us have played with hula hoops as children, we would have a hard time explaining exactly how we kept the hoop from falling off our waists, arms, legs, or (for the more talented amongst us) even noses. Is it better to move around very fast, very slowly, or somewhere in between? Does the size of the hula hoop matter? Do you have to make different types of movements to keep the hula hoop around your waist than, say, around your arm? Can you get a hula hoop to stay up around anything, or can it only twirl around certain shapes? We say that a hula hoop twirling about a rotating body is in *equilibrium* when the hula hoop will keep twirling around the same place on the body as long as the body doesn't change how it is moving. In other words, if you have found an equilibrium point for your hula hoop and you keep doing exactly what you've been doing to get it there, the hula hoop will not fall.

But what happens if, once the hula hoop is twirling around you in equilibrium, someone comes along and tries to knock the hoop off you? How easy would it be for someone to knock it off? Generally, we want to know how easy it is to find equilibria, and, once you've found them, how easy is it to remain in equilibrium. This is referred to as the *stability* of the equilibria.

To answer the questions I just asked, we need to model the motion of a hula hoop about a rotating body and use this model to deduce information about the system's equilibria and their stability. That is our current research problem. We will use constraint equations and

Lagrangian dynamics to model a hula hoop twirling about a rotating body. I will explain what this means and how to do this in section 3.

## 2 A Simple Model

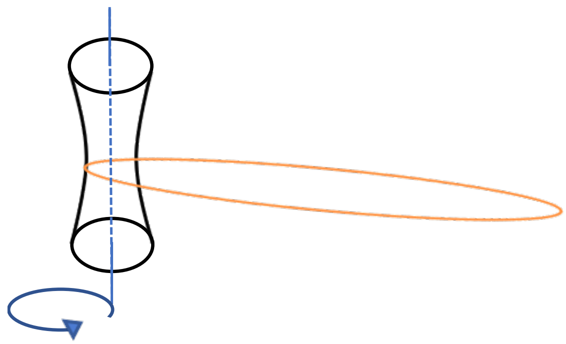

Figure 1: This is a simple schematic of a hoop twirling about a rotating upright rigid body. Here, the rotating body is a hyperboloid, and the hula hoop (in orange) is twirling about it.

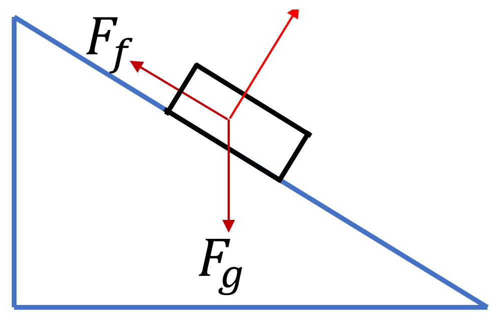


Figure 2: This is a block on an incline. The block feels gravity, friction, and the normal force.

Modeling a hula hoop twirling about a real person's body is complicated. People's bodies are all unique and they have complex shapes. People are also slightly squishy, not rigid. And the motions that a person makes to keep the hula hoop up won't be exactly identical every time. These things are complicated to study, and so we need to come up with a simpler model if we want to study that model analytically rather than numerically. Analytical work allows us to study an exact model rather than a numerical approximation to a model. Numerical work comes with some innate approximations that cannot be removed. Depending on the system studied, these approximations could have a great impact on the system's dynamics. By simplifying the model such that it can be solved analytically, I can choose exactly which approximations and simplifications I wish to make, and so ensure that they are appropriate for the system I am studying. This will allow us greater freedom to identify the most important effects that keep the hoop from falling, and hence better understand the fundamental relationships and quantities that describe the hoop's motion. In short, analytical work will tell us *why* and *when* a hula-hoop stays up, whereas numerical work can tell us *whether* it stays up for a specific case. That is why we want a model that ignores things that are too complicated while making sure that the model remains sufficiently realistic for us to make conclusions about real humans.

A simple model is as follows: an upright rigid body such as a hyperboloid, cylinder or cone, rotating in such a way that the body moves in a circle with known frequency and without spinning about itself. A hoop is then spun about the body. This is illustrated in figure (1). The hoop could fall off the body, fly off the top of the body, or find an equilibrium point. We want to use constraint equations and Lagrangian dynamics to find the path of the hula hoop as it twirls about a rotating body. In the following section, I will explain what this means.

A possible future way to check if this model matches reality would be to ask people to hula hoop in a lab, where their motion (as well as the hoop's) could be tracked. We could then compare a real hula hooper's motions to our model's predictions for equilibria and their stability, and see how well they match.

# 3 Lagrangian mechanics and Constraint equations

## 3.1 Newtonian mechanics (and it's limitations)

In this section, I'll provide the reader with a brief introduction into Lagrangian mechanics and the use of constraint equations through examples. I will also explain why the constraints on the hula hoop make the system challenging to solve. If you have taken an elementary physics course, you are familiar with Newtonian mechanics and free body diagrams, also called force-balance diagrams. Force-balance diagrams allow us to calculate, for example, the motion of a block sliding down an incline. To do this, we first identify the forces at work: there is the gravitational force pointing straight down, the normal force pointing normal to the incline, and the frictional force pointing up along the incline, as in figure (2). Then, using a bit of geometry, we can calculate the net force, which will be down along the incline. Finally, we use Newton's 2$^{\text{nd}}$ law. This law tells us that $F(t) = ma(t)$, where $F(t)$ is the net force we've just calculated, $m$ is the mass of the block, and $a(t)$ is the acceleration of the block. Once we have $F(t)$ and $m$, we can solve for $a(t)$. We then have an ordinary differential equation (ODE) that we may or may not be able to solve. In the case of a block on an incline, we can solve the ODE and obtain the path of the block down the incline. However, for systems that are less simple than the one just described, we may end up with an ODE that cannot be solved analytically. Take, for example, a bead on a curved wire. The normal force points normal to the wire, and so its direction changes as the bead slides along the wire. If the wire were straight, the normal force would be easy to calculate, but if it curves in a complicated way, the normal force could be difficult to find and write down. This makes the use of a force-balance diagram and Newton's second law more complicated. Writing down $F(t)$ will be more difficult, and it might not be possible to solve the resulting ODE analytically.

## 3.2 Constraint Equations

Note that for both the block on the incline example and the bead on the wire example, the normal force is a *constraint force*. That is, it constrains the block to stay on the incline, and the bead to stay on the wire. Without this force, both the bead and the block would have simply fallen straight down under the force of gravity. In both cases, this was also the force that, due to its time dependence, made the Newtonian mechanics calculation difficult. We need another method for finding the path of objects in such cases. Fortunately, there exists another method: Lagrangian mechanics. I will illustrate the key ideas of Lagrangian mechanics through a worked example.

Imagine a bead on a parabolic wire, as shown in figure (3). As long as the bead's position satisfies the following *constraint equation* (1), the bead will stay on the wire:

$$g(x(t), z(t)) = z(t) - x(t)^2 = 0, \tag{1}$$

where the bead's position at time $t$ is given by $(x(t), z(t))$, and $g(x(t), z(t))$ is known as the constraint function. This is called a *constraint equation* because it describes the constraint that the bead remain on the wire. One way to understand equation (1) is to read from it that the bead's position must obey the equation of the wire's parabola.

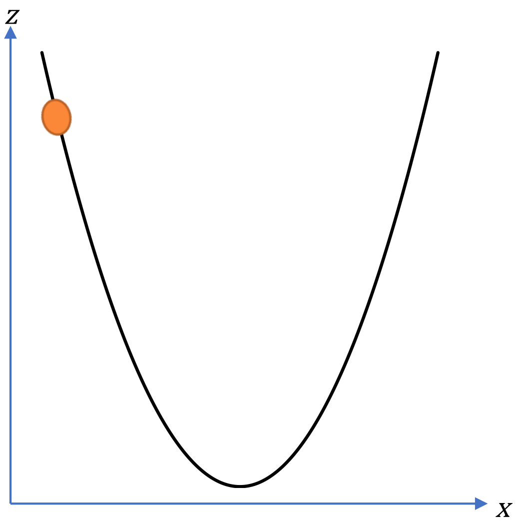


Figure 3: A bead (orange oval) on a parabolic wire. The bead's position is given by $(x(t), z(t)) = (x(t), x^2(t))$. This is an example of a holonomic system.

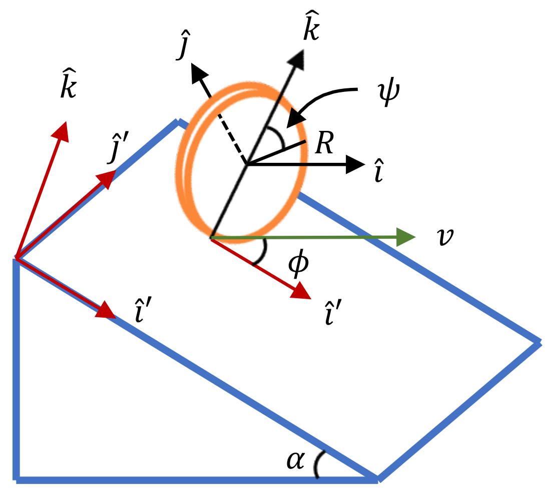


Figure 4: A coin of radius $R$ and mass $M$ rolls down an incline. The coin is constrained to stay upright, as shown in the diagram, but can move freely by rolling (which is measured by $\psi$) or twisting (measured by $\phi$). The axis $\hat{\boldsymbol{\imath}}$ points in the direction in which the coin rolls, and $\hat{\boldsymbol{\jmath}}$ is perpendicular to the surface of the coin. When the coin rolls, it rotates about $\hat{\boldsymbol{\jmath}}$. Note that the change in $\psi$ is equal to the distance rolled divided by $R$. This is an example of a non-holonomic system.

We can make use of this equation to find the path of the bead along the wire. The first thing we will need is a quantity called the *Lagrangian*. The Lagrangian is found by subtracting the potential energy of the system from the kinetic energy. In other words,

$$L(x, z, \dot{x}, \dot{z}, t) = \frac{1}{2}m(\dot{x}(t)^2 + \dot{z}(t)^2) - \frac{1}{2}mz(t). \tag{2}$$

We recall that along a parabolic wire, $z(t) = x^2(t)$. We can substitute this into the

Lagrangian, as follows:

$$L(x, \dot{x}, t) = \frac{1}{2}m(\dot{x}(t)^2 + 2x(t)\dot{x}(t)^2) - \frac{1}{2}mx(t)^2. \tag{3}$$

Then, the path of the bead along the wire is given by the solution of the following equations, called the *Euler-Lagrange equation* for this system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0. \tag{4}$$

If we substitute the Lagrangian from equation (2) into the Euler-Lagrange equation, we obtain the following ODE:

$$\ddot{x}(1 + 2x) + \dot{x}^2 + x = 0. \tag{5}$$

The solution of this ODE is the path of the bead on the parabolic wire.

In this example, we used the constraint equation to re-write the equations describing the bead's motion, thus allowing us to phrase the problem with fewer degrees of freedom. We noticed that we could re-write both $x$ and $z$ in terms of $x$ alone. Thus we were left with only one ODE to solve. There is another way in which we could have used the constraint equation and Euler-Lagrange equations to find the path of the bead along the parabolic wire. Instead of substituting the change of variables $z(t) = x(t)^2$ from the constraint equation directly into the Lagrangian, we could have written out the following system of equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \lambda\left(\frac{\partial g}{\partial x}\right) \tag{6a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = \lambda\left(\frac{\partial g}{\partial z}\right) \tag{6b}$$

$$g(x(t), z(t)) = 0, \tag{6c}$$

where $g(x(t), z(t))$ is the constraint function in equation (1), and $\lambda$ is a constant called a Lagrange multiplier. This system of equations is called the *constrained Euler-Lagrange equations*. Here, we are treating the constraint equation as an extra equation, to be solved in conjunction with the Euler-Lagrange equations. We use the method of Lagrange multipliers to write down the constrained Euler-Lagrange equations. If we substitute the Lagrangian from equation (2) and the constraint function from equation (1) into this system of equations, then solve the system, we will obtain the same solution as if we were to solve the ODE in equation (5). These are two ways in which Lagrangian mechanics can be used to find the path of objects in a constrained system.

To summarize, constraint equations can suggest a choice of coordinates and be incorporated into the Lagrangian through this choice of coordinates. They can also become part of the constrained Euler-Lagrange equations. Will one or both of these options always be available to us, for every conceivable constraint equation? In the following section, we will explore two broad classes of constraints.

## 3.3 Holonomic and Non-Holonomic Constraints

Constraints can be divided into two categories: holonomic constraints and non-holonomic constraints. Holonomic constraints, also called geometric constraints, limit where an object can be. Constraints of this type can be written entirely in terms of position and angle coordinates. Some examples include our earlier constraints that the block remain on the incline, or that the bead remain on the wire. This can be contrasted to a non-holonomic constraint for which it is impossible to write the constraints as a function of position and angle only. Rather, the constraint equations must depend on velocity and angular velocity. This distinction has a significant impact when looking to find the path of an object using Lagrangian mechanics, and so it is worth spending time to understand. I will now describe an example of a holonomic and of a non-holonomic system that are easy to visualize, in order to understand the fundamental difference between the two.

Let's contrast our earlier examples to a non-holonomic system. A hula hoop twirling about a cylindrical body is a non-holonomic system. However, I will consider a simpler problem in this subsection: rather than the hula hoop, let's imagine a twirling penny of radius $R$ rolling down an incline, as in figure (4). I will focus on this example rather than the hula hoop because it has fewer variables to keep track of, and is consequently better for illustrative purposes. If we force the coin to stay upright, we only need two position coordinates and two angle coordinates to completely describe it. Let us call these variables $\boldsymbol{q} = (q_1, q_2, q_3, q_4) = (x, y, \psi, \phi)$, as in figure (4). The position coordinates $x$ and $y$ describe the distance travelled up or down the incline and the distance travelled across the width of the incline, respectively. The angle $\psi$ describes the rotation of the coin about the axis normal to the coin. In other words, if you were to put a stick through the center of the coin and turn the stick, this would change the value of $\psi$. The angle $\phi$ gives the direction the coin is pointing in. For example, if $\phi = 0$, the coin is pointing straight down the incline (along $\hat{\boldsymbol{i}}'$). The constraint on this system is the no-slip constraint, which, when projected onto $\hat{\boldsymbol{i}}$, the principal axis in which direction the coin is rolling, and $\hat{\boldsymbol{j}}$, the normal to the surface of the coin, gives the following constraint functions:

$$g_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \dot{x}\cos(\phi) + \dot{y}\sin(\phi) - R\dot{\psi} = 0, \quad g_2(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \dot{x}\sin(\phi) - \dot{y}\cos(\phi) = 0. \tag{7}$$

Note that these constraints contain velocities. How do we know that they *must* contain velocities, and that there is no clever trick that would allow us to re-write the constraints in equation (7) without any velocities? To answer this question, let's recall that in a holonomic system, it is possible to re-write the system in such a way as to reduce the total number of degrees of freedom. In the case of the bead on the wire, the constraint equation (1) gave us $z = x^2$. So, do we truly need each of the four variables $(x, y, \psi, \phi)$ to describe the coin's twisting and rolling motion? We need $x$ and $y$, since the coin can reach any point on the incline through some combination of twisting and rolling. With enough initial velocity, you can even make the coin move up the plane. Then, we note that the coin can attain any angle $\phi$ at any values of $x$ and $y$: simply imagine holding the coin in place and twisting it to the desired angle. So we have established that we need 3 out of the 4 variables given in the paragraph above. Can the coin achieve any value of $\psi$ at a given $(x, y, \phi)$?

Suppose we place the coin at the point $(q_1, q_2, q_3, q_4) = (x_0, y_0, \psi_0, \phi_0)$, and that we want to move the coin without slipping such that it ends up with the same $x, y, \phi$ values as before, but with $\psi = \psi_1$ instead. To do this, we recall that the change in $\psi$ is equal to the distance rolled divided by $R$. So we need to move the coin along a closed path of length $R \cdot (\psi_1 - \psi_0 + 2k\pi)$. A circle of radius $(\psi_1 - \psi_0 + 2k\pi)/2\pi \cdot R$ works, with $k \in \mathbb{Z}$. So, to get the coin into the position $(x_0, y_0, \psi_1, \phi_0)$, we roll the coin in a circle of radius $(\psi_1 - \psi_0 + 2k\pi)/2\pi \cdot R$, then adjust $\phi$ as needed.

We've just shown that we can attain any state $(x, y, \psi, \phi)$ using only rolling and twisting motions, i.e. without slipping. Therefore, the no-slip constraint cannot be used to decrease the number of variables needed to fully describe the system. We can now conclude with certainty that the coin on the incline constitutes a non-holonomic system.

## 3.4 The Euler-Lagrange equations for Non-Holonomic systems

Non-holonomy has significant consequences when looking to solve a system using Lagrangian dynamics. Our typical way of applying the Euler-Lagrange equations to a constrained system breaks down when the constraints are non-holonomic, and we must find a different way to solve [1] [2]. Fortunately, the principle from which the Euler-Lagrange equations are derived, called the *D'Alembert-Lagrange principle* or *principle of virtual work*, remains valid and can be used on systems with non-holonomic constraints [1]. This principle tells us that constraint forces do no work. From this, it is possible to work out the Euler-Lagrange equations for non-holonomic systems, which I will now state.

Let our system have $c$ general non-holonomic constraints of the form $g_k(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0$, where $k = 1, 2, \ldots c$. Then, for a particle described by the $n$ generalized coordinates $\boldsymbol{q}(t) = (q_1, \ldots, q_n)$ and $\dot{\boldsymbol{q}}(t) = (\dot{q}_1, \ldots, \dot{q}_n)$, the path $\boldsymbol{q}(t)$ is given by the solution of the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} + \sum_k \lambda_k \left(\frac{\partial g_k}{\partial \dot{q}_j}\right), \tag{8}$$

where $\lambda_k$ are Lagrange multipliers, $Q_j^{NP}$ are any forces not included in the Lagrangian, and $L$ is the Lagrangian [1].

To understand how to apply equation (8) to a concrete non-holonomic system, we recall our earlier system: a twirling coin of radius $R$ and mass $M$ rolling down an incline. There are fewer variables to worry about in this problem than for the hula hoop problem, and so we can focus on understanding and applying equation (8). Let us take as generalized coordinates $(q_1, q_2, q_3, q_4) = (x, y, \psi, \phi)$, as before. Then, if the incline is at an angle $\alpha$ to the horizontal, the Lagrangian is given by:

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_2\dot{\psi}^2 + \frac{1}{2}I_3\dot{\phi}^2 + Mgx\sin\alpha, \tag{9}$$

where $I_2 = \frac{1}{2}MR^2$ and $I_3 = \frac{1}{4}MR^2$ are the moments of inertia of the body about two of the coin's principal axes: $\hat{\boldsymbol{j}}$, the axis normal to the surface of the coin, and $\hat{\boldsymbol{k}}$, the axis normal to the surface of the incline, respectively.

The constraints on this system are as before:

$$g_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \dot{x}\cos(\phi) + \dot{y}\sin(\phi) - R\dot{\psi} = 0, \;\; g_2(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \dot{x}\sin(\phi) - \dot{y}\cos(\phi) = 0. \tag{10}$$

We now have $L$ from equation (9) and $g_k, k = 1, 2$ from equation (10). All the forces in this system are potential forces, so $Q_j^{NP} = 0$ for this system. We can then use equation (8) to find $q(t)$ [1]. If we substitute in equations (9) and (10) into (8), we obtain the following system of equations:

$$M\ddot{x} = -Mg\sin\alpha + \lambda_1\cos(\phi) + \lambda_2\sin(\phi) \tag{11a}$$

$$M\ddot{y} = \lambda_1\sin(\phi) - \lambda_2\cos(\phi) \tag{11b}$$

$$I_2\ddot{\psi} = \lambda_1\sin\phi \tag{11c}$$

$$I_3\ddot{\phi} = 0. \tag{11d}$$

Equations (10) and (11) together give us six equations, and there are six unknowns: two Lagrange multipliers and four coordinates. This system can be solved by first eliminating the $\lambda_k$ from the equations analytically, then using the explicit trapezoid method to solve the resulting four ODEs numerically.

## 4 Conclusion

We have constructed a model for a hula hoop twirling about a rotating rigid body and seen that the constraints on the hoop are non-holonomic. We then learned that the path of objects moving under non-holonomic constraints can be found by solving a modified version of the constrained Euler-Lagrange equations. We also practiced using these equations by finding the path of a coin rolling down an incline. Going forward, we will use these equations to find the path traced out by a hula hoop as it twirls about a body, as we did for the coin. We can do this for various bodies and study the hoop's equilibria and their stability for these bodies. We will start by considering a cylindrical body, then look at cones and hyperboloids.

However, it will be trickier to find the path of the hula hoop than it was to find the path of the coin, as more coordinates are needed to describe the hula hoop. To deal with this, we plan to use quaternions as generalized coordinates to describe the position and inclination of the hula hoop as it journeys about the hula hooper's body. However, the Lagrangian for this system is explicitly time-dependent, which introduces new difficulties. Once these difficulties have been addressed and the simple model described in section 2 has been solved, we will look at the effect of the body's path and inclination: what happens if the body moves in an oval, or a flower shape, rather than a circle? And what if the body is allowed to wobble, rather than being always upright? What if it is slanted, or has a funny shape? We hope to eventually compare our results to the behavior of a hula hoop when used by a real hula hooper.

## References


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[2] Ian R Gatland, *Nonholonomic constraints : A test case*, American Journal of Physics **72** (2004), no. 941.